

# Equations for spinning test particles in equatorial orbits when they are orbiting in a weak rotating field

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## Abstract

This paper formulates, via the Mathisson - Papapetrou - Dixon equations, the system of equations for a test particle with spin when it is orbiting a weak Kerr metric. We shall restrict ourselves to the case of circular orbits with the purpose of comparing our results with the results of the literature. In particular, we solve the set of equations of motion for the case of circular trajectories both spinless and spinning test particles around rotating bodies in equatorial plane. The results obtained are an important guideline for the study of the effects of the particles with spin in rotating gravitational fields such as Gravitomagnetism Effects or gravitational waves.

## 1 Introduction

Presently, there exists an interest in the study of the effects of the spin in the trajectory of test particles in rotating gravitational fields. The importance of this topic increases when dealing with phenomena of astrophysics such as accretion discs in rotating black holes [1], Gravitomagnetism Effects [2] or gravitational waves induced by spinning particles orbiting a rotating black hole [3]. Therefore, we shall work the equations of motion for test particles in a weak Kerr metric which will be integrated numerically in the particular case when the test particles are orbiting circularly with the purpose of studying the effects of spin in the trajectories of test particles in rotating gravitational fields.

The motion of particles in a gravitational field is given by the geodesics equation. The solution to this equation depends on the problem, and therefore there are different methods for its solution [4] [5]. Basically, we take two cases in motion of test particles in a gravitational field of a rotating mass. The first case describes the trajectory of a spinless test particle, and the second one the trajectory of a spinning test particle in a weak Kerr metric. For the second case a representation is used that does not include the third-order derivatives of the

coordinates, and yields the equations of motion for a spinning test particle in a gravitational field without any restrictions on its velocity and spin orientations [6].

For the first case, authors as Tanaka *et al.* [1] yield the set of equations of motion for orbiting spinless test particles. In this case the equations of motion for the spinless test particle are considered both in the equatorial [7], [8], [9], and the non-equatorial plane [8], [10], [11] (Kheng, L., Perng, S., Sze Jackson, T.: Massive Particle Orbits Around Kerr Black Holes. Unpublished).

For the study of test particles in a rotating field, some authors have solved the equations of motion for spinless and spinning test particles in the particular case of circular orbits in the equatorial plane of a Kerr metric [1], [7], [12], [13], [14], [15], [16], [17]. In addition, Plyastko, R. *et al.* yield the full set of Mathisson-Papapetrou-Dixon Equations (MPD equations) for spinning test particles in the Kerr gravitational field [6]. These authors integrate numerically the MPD equations for the case of the Schwarzschild metric. In this paper, we use the method of MPD Equations given by Plyastko, R. *et al.* for calculating the trajectories of spinless and spinning test particles in equatorial planes for circular orbits, i.e., constant radius in a weak Kerr metric. In the literature, there are not works that study via MPD equations the trajectories of spinning test particles in weak fields.

With the purpose to prove the equations of motion that we worked, we shall solve numerically the set of equations of motion obtained via MPD Equations in the case when the spinless test particle is in the equatorial plane and will compare the results with works that involve astronomy, especially the study of satellites which orbit around the Earth. We take the same initial conditions in the two cases for describing the trajectory both a spinless particle and a spinning particle in a weak Kerr metric. Then, we compare the cartesian coordinates  $(x, y, z)$  for the trajectory of two particles that travel in the same orbit but in opposite directions. We shall take both for a spinless test particle and for a spinning test particle orbiting in a weak Kerr field.

This work is organized as follows. In Section 2 we give a brief introduction to the MPD Equations that work the set of equations of motion for test particles both spinless and spinning in a rotating gravitational field. From the MPD equations of motion, we yield the equations of motion for spinless and spinning test particles and will study the spinless test particles. Also, we will give the set of MPD equations given by Plyatsko *et al.* [6] in a schematic form for working the case of a weak Kerr metric. In Section 3, we present the Gravitomagnetic Clock Effect in order to prove our set of equations for spinless and spinning test particles. Then, in Section 4, we make a numerical comparison for spinless and spinning test particle via MPD equations in the equatorial plane. We take the initial values from a satellite that is orbiting around on the Earth; then, we substitute these values in the MPD equations both for spinless particles and for spinning particles, and finally we make a numerical comparison of the trajectory in cartesian coordinates for two particles that travel in the same orbit, but in opposite directions. In the last section, the conclusions and some future works are formulated in order to describe spinning test particles in a weak Kerr metric.

## 2 Brief introduction to the Mathisson-Papapetrou-Dixon Equations

In general the MPD equations [18], [19], [20] are given by the dynamics of extended bodies in the general theory of relativity which includes any gravitational background. For the solution of our problem, we take the case of a distribution of mass ( $m$ ) with a spin tensor ( $S^{\rho\sigma}$ ) around a rotating central source ( $M$ ) which has a metric tensor  $g_{\mu\nu}$ . These equations of motion for a spinning test particle are obtained in terms of an expansion that depends on the derivatives of the metric and the multipole moments of the energy-momentum tensor ( $T^{\mu\nu}$ ) [20] which describe the motion of an extended body. In this work, we shall take a body sufficiently small so that all higher multipoles can be neglected. According to this restriction the MPD equations are given by

$$\frac{D}{ds} \left( mu^\lambda + u_\lambda \frac{DS^{\lambda\mu}}{ds} \right) = -\frac{1}{2} u^\pi S^{\rho\sigma} R_{\pi\rho\sigma}^\lambda, \quad (1)$$

$$\frac{D}{ds} S^{\mu\nu} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{ds} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{ds} = 0, \quad (2)$$

where  $D/ds$  means the covariant derivative, and the antisymmetric tensor  $S^{\mu\nu}$  are the linear and spin angular momenta, respectively.  $R_{\pi\rho\sigma}^\lambda$  is the curvature tensor, and  $u^\mu = dz^\mu/ds$ . But we do not have the evolution equation for  $u^\mu$  and it is necessary to single out the center of mass which determines the world line as a representing path and specifies a point about which the momentum and spin of the particle are calculated. This world line can be determined from physical considerations [21]. In general, two conditions are usually imposed. The Mathisson-Pirani supplementary condition is [18] [22]

$$u_\sigma S^{\mu\sigma} = 0 \quad (3)$$

and the Tulczyjew-Dixon condition [20]

$$p_\sigma S^{\mu\sigma} = 0 \quad (4)$$

where

$$p^\sigma = mu^\sigma + u_\lambda \frac{DS^{\sigma\lambda}}{ds} \quad (5)$$

is the four momentum.

For to obtain the set of MPD equations, we take the MP condition (3) which has three independent relationships between  $S^{\mu\sigma}$  and  $u_\sigma$ . By this condition  $S^{i4}$  is given by

$$S^{i4} = \frac{u_k}{u_4} S^{ki} \quad (6)$$

with this expression we can deal the independent components  $S^{ik}$ . Sometimes it is more convenient the vector spin which is given by  $S_i = \frac{1}{2u_4} \sqrt{-g} \epsilon_{ikl} S^{kl}$ , where  $\epsilon_{ikl}$  is the spatial Lévi-Civita symbol.

On the other hand, when the space-time admits a Killing vector  $\xi^\nu$ , there exists a property that includes the covariant derivative and the spin tensor, which gives a constant and is given by the expression [23]

$$p^\nu \xi_\nu + \frac{1}{2} \xi_{\nu,\mu} S^{\nu\mu} = \text{constant}, \quad (7)$$

where  $p^\nu$  is the linear momentum,  $\xi_{\nu,\mu}$  is the covariant derivative of the Killing vector, and  $S^{\nu\mu}$  is the spin tensor of the particle. In the case of the Kerr metric, there are two Killing vectors, owing to its stationary and axisymmetric nature. In consequence, Eq. (7) yields two constants of motion:  $E$  is the total energy and  $L$  is the component of its angular momentum along the axis of symmetry [24].

## 2.1 MPD Equations for a spinning test particle in a metric of rotating body

Given that the spinning body test is sufficiently small in regard to the characteristic length the equations of motion (Eqs. 1 and 2) are reduced to the case when the test particles are orbiting a metric of rotating body. Then, we will give the equations of motion for the case of a spinning test particle for a weak Kerr metric (Appendix A).

First of all, we take the paper by R.M. Plyatsko *et al.* [6] for obtaining the full set of the exact MPD equations for the motion of a spinning test particle in the Kerr field if the MP condition (3) is taken into account and obtain a general scheme for the set of equations of motion for a spinning test particle in a rotating field. Plyatsko *et al.* use the dimensionless quantities  $y_i$  with particle's coordinates by

$$y_1 = \frac{r}{M}, \quad y_2 = \theta, \quad y_3 = \varphi, \quad y_4 = \frac{t}{M} \quad (8)$$

for its 4-velocity

$$y_5 = u^1, \quad y_6 = Mu^2, \quad y_7 = Mu^3, \quad y_8 = u^4 \quad (9)$$

and the spin components [25]

$$y_9 = \frac{S_1}{mM}, \quad y_{10} = \frac{S_2}{mM^2}, \quad y_{11} = \frac{S_3}{mM^2} \quad (10)$$

In addition, they introduce another dimensionless quantities with regard to the proper time  $s$  and the constant of motion  $E, J_z$

$$x = \frac{s}{M}, \quad \hat{E} = \frac{E}{m}, \quad \hat{J} = \frac{J_z}{mM} \quad (11)$$

The set of the MPD equations for a spinning particle in the Kerr field is given by eleven equations. The first four equations are

$$\dot{y}_1 = y_5, \quad \dot{y}_2 = y_6, \quad \dot{y}_3 = y_7, \quad \dot{y}_4 = y_8 \quad (12)$$

where a dot denotes the usual derivative with respect to  $x$ .

The fifth equation is given by the first three equations of (1) with the indexes  $\lambda = 1, 2, 3$ . The result is multiplied by  $S_1, S_2, S_3$  and with the MP condition (3) we have the relationship:  $S^{i4} = \frac{u_i}{u_4} S^{ki}$  and  $S_i = \frac{1}{2u_4} \sqrt{-g} \varepsilon_{ikl} S^{kl}$ , we obtain

$$m S_i \frac{Du^i}{ds} = -\frac{1}{2} u^\pi S^{\rho\sigma} S_j R_{\pi\rho\sigma}^j \quad (13)$$

which can be written as

$$y_9 \dot{y}_5 + y_{10} \dot{y}_6 + y_{11} \dot{y}_7 = A - y_9 Q_1 - y_{10} Q_2 - y_{11} Q_3 \quad (14)$$

where

$$Q_i = \Gamma_{\mu\nu}^i u^\mu u^\nu, \quad A = \frac{u^\pi}{\sqrt{-g}} u_4 \epsilon^{i\rho\sigma} S_i S_j R_{\pi\rho\sigma}^j \quad (15)$$

The sixth equation is given by

$$u_\nu \frac{Du^\nu}{ds} = 0 \quad (16)$$

which can be written as

$$p_1 \dot{y}_5 + p_2 \dot{y}_6 + p_3 \dot{y}_7 + p_4 \dot{y}_8 = -p_1 Q_1 - p_2 Q_2 - p_3 Q_3 - p_4 Q_4 \quad (17)$$

where

$$p_\alpha = u_\alpha = g_{\mu\alpha} u^\mu \quad (18)$$

The seventh equation is given by

$$E = P_4 - \frac{1}{2} g_{4\mu,\nu} S^{\mu\nu} \quad (19)$$

which can be written as

$$c_1 \dot{y}_5 + c_2 \dot{y}_6 + c_3 \dot{y}_7 = C - c_1 Q_1 - c_2 Q_2 - c_3 Q_3 + \hat{E} \quad (20)$$

where

$$d = \frac{1}{\sqrt{-g}}$$

$$\begin{aligned} c_1 &= -dg_{11}g_{22}g_{44}u^2S_3 - d(g_{34}^2 - g_{33}g_{44})g_{11}u^3S_2 \\ c_2 &= dg_{11}g_{22}g_{44}u^1S_3 + d(g_{34}^2 - g_{33}g_{44})g_{22}u^3S_1 \\ c_3 &= d(g_{34}^2 - g_{33}g_{44})g_{11}u^1S_2 - d(g_{34}^2 - g_{33}g_{44})g_{22}u^2S_1 \end{aligned} \quad (21)$$

$$C = g_{44}u^4 - dg_{44}u^4g_{43,2}S_1 + d(g_{44}u^4g_{43,1} - g_{33}u^3g_{44,1})S_2 + dg_{22}u^2g_{44,1}S_3 \quad (22)$$

The eighth equation is given by

$$J_z = -P_3 + \frac{1}{2}g_{3\mu,\nu}S^{\mu\nu} \quad (23)$$

which can be written as

$$d_1\dot{y}_5 + d_2\dot{y}_6 + d_3\dot{y}_8 = D - d_1Q_1 - d_2Q_2 - d_3Q_4 - \hat{J} \quad (24)$$

where

$$\begin{aligned} d_1 &= -dg_{11}g_{22}g_{34}u^2S_3 + dg_{11}g_{33}g_{34}u^3S_2 + dg_{11}g_{34}^2u^4S_2 - dg_{11}g_{33}g_{44}u^4S_2 \\ d_2 &= -dg_{11}g_{22}g_{34}u^1S_3 - dg_{22}g_{33}g_{34}u^3S_1 - dg_{22}g_{34}^2u^4S_1 + dg_{22}g_{33}g_{44}u^4S_1 \\ d_3 &= -dg_{11}g_{34}^2u^1S_2 + dg_{22}g_{34}^2u^2S_1 + dg_{22}g_{33}g_{44}u^2S_1 - dg_{11}g_{33}g_{34}u^1S_2 \end{aligned} \quad (25)$$

$$\begin{aligned} D &= g_{33}u^3 - dg_{22}u^2g_{34,2}S_1 + d(g_{44}u^4g_{33,1} + g_{11}u^1g_{34,1} - g_{33}u^3g_{34,1})S_2 \\ &\quad - dg_{11}u^1g_{34,1}S_3 \end{aligned} \quad (26)$$

Finally, the last three equations are given by

$$u^4\dot{S}_i + 2\left(\dot{u}_{[4}u_{i]} - u^\pi u_\rho \Gamma_{\pi[4}^\rho u_{i]}\right)S_k u^k + 2S_n \Gamma_{\pi[4}^n u_{i]}u^\pi = 0 \quad (27)$$

which give the derivatives of three components of vector spin ( $\dot{S}_i$ ):  $\dot{y}_9$ ,  $\dot{y}_{10}$  and  $\dot{y}_{11}$ .

After achieving the system of equations of motion for spinning test particles, we numerically solve it. We used the fourth-order Runge Kutta method. First we take the case when a test particle is orbiting far away from the central source and is in the equatorial plane. For our numerical calculations, we take the parameters both of the central source and the test particle such as the radio, the energy, the angular momentum and the components tangential and radial of the four-velocity ( $u^\mu$ ). We calculate the orbit of a test particle both spinless and spinning a weak Kerr metric in cartesian coordinates ( $x, y, z$ ). Then, we make a comparison of the time that a test particle takes to do a lap in the two cases and give some conclusions.

## 2.2 Equations of motion for a spinning test particle in a weak Kerr metric

In the last section, we obtained the general scheme for the set of equations of motion of a spinning test particle in the gravitational field of a rotating body. Now, we yield the set of equations for the case of a spinning test particle in the equatorial plane of a weak metric Kerr (Appendix A). This set of equations is given by

$$r'[s] = \frac{dr}{ds}; \quad \theta'[s] = \frac{d\theta}{ds} = 0; \quad \varphi'[s] = \frac{d\varphi}{ds}; \quad t[s] = \frac{dt}{ds} \quad (28)$$

$$\frac{d^2 r}{ds^2} = \left( \frac{c_3 d_3}{R_1} \right) \left( \frac{R_2}{c_3} + \frac{R_3}{d_3} + R_4 \right) \quad (29)$$

$$\frac{d^2 \varphi}{ds^2} = \left( \frac{-c_1 d_3}{R_1} \right) \left( \frac{R_2}{c_3} + \frac{R_3}{d_3} + R_4 \right) + \frac{\left( B - c_1 Q_1 + \hat{E} \right)}{c_3} - Q_3 \quad (30)$$

$$\frac{d^2 t}{ds^2} = \left( \frac{-d_1 c_3}{R_1} \right) \left( \frac{R_2}{c_3} + \frac{R_3}{d_3} + R_4 \right) + \frac{\left( F - d_1 Q_1 - \hat{J} \right)}{d_3} \quad (31)$$

$$z := (r[s])^2; \quad q := r[s] (r[s] - 2); \quad \psi := (r[s])^2$$

$$\eta := 3 (r[s])^2; \quad \chi := (r[s])^2; \quad \xi := (r[s])^2$$

$$p := 2\alpha \left( \sin \left( \frac{\pi}{2} \right) \right)^2 r[s] \frac{d\varphi}{ds} + \left( (r[s])^2 - 2r[s] \right) \frac{dt}{ds};$$

$$p_1 := - \left( 1 - \frac{2}{r[s]} \right)^{-1} \frac{dr}{ds}; \quad p_2 := 0;$$

$$p_3 := \frac{2\alpha \left( \sin \left( \frac{\pi}{2} \right) \right)^2 r[s] \frac{dt}{ds} - \left( \sin \left( \frac{\pi}{2} \right) \right)^2 (r[s])^4 \frac{d\varphi}{ds}}{(r[s])^2};$$

$$p_4 := \frac{2\alpha \left( \sin \left( \frac{\pi}{2} \right) \right)^2 r[s] \frac{d\varphi}{ds} + \left( (r[s])^2 - 2r[s] \right) \frac{dt}{ds}}{(r[s])^2}$$

$$c_1 := S_2 \sin \left( \frac{\pi}{2} \right) \frac{d\varphi}{ds}; \quad c_2 := 0; \quad c_3 := -S_2 \sin \left( \frac{\pi}{2} \right) \frac{dr}{ds}$$

$$d_1 := -S_2 \sin \left( \frac{\pi}{2} \right) \frac{dt}{ds}; \quad d_2 := 0; \quad d_3 := S_2 \sin \left( \frac{\pi}{2} \right) \frac{dr}{ds}$$

$$Q_1 := \frac{((r[s] - 2) - r[s] + 2)}{(r[s]) (r[s] - 2) + \alpha^2} \left( \frac{dr}{ds} \right)^2 - \left( \sin \left( \frac{\pi}{2} \right) \right)^2 (r[s] - 2) \left( \frac{d\varphi}{ds} \right)^2 \\ + \frac{(r[s] - 2)}{(r[s])^3} \left( \frac{dt}{ds} \right)^2 - \frac{2\alpha \left( \sin \left( \frac{\pi}{2} \right) \right)^2 (r[s] - 2)}{(r[s])^3} \frac{d\varphi}{ds} \frac{dt}{ds}$$

$$\begin{aligned}
Q_2 &:= 0 \\
Q_3 &:= \frac{2}{(r[s])} \frac{dr}{ds} \frac{d\varphi}{ds} + \frac{2\alpha}{(r[s])^3 (r[s] - 2)} \frac{dr}{ds} \frac{dt}{ds} \\
Q_4 &:= -\frac{6\alpha}{(r[s]) (r[s] - 2)} \frac{dr}{ds} \frac{d\varphi}{ds} + \frac{2}{(r[s]) (r[s] - 2)} \frac{dr}{ds} \frac{dt}{ds} \\
B &:= -\left(1 - \frac{2}{(r[s])}\right) \frac{dt}{ds} - \frac{2\alpha \left(\sin\left(\frac{\pi}{2}\right)\right)^2}{(r[s])^2} \frac{dr}{ds} \frac{d\varphi}{ds} + \frac{S_2 \sin\left(\frac{\pi}{2}\right)}{(r[s])^2} \frac{d\varphi}{ds} - \frac{\alpha S_2 \sin\left(\frac{\pi}{2}\right)}{(r[s])^4} \frac{dt}{ds} \\
F &:= -\frac{3\alpha S_2 \left(\sin\left(\frac{\pi}{2}\right)\right)^3}{(r[s])^2} \frac{d\varphi}{ds} - \frac{S_2 ((r[s]) - 2)}{(r[s])^2} \frac{dt}{ds} \\
&\quad + \frac{\left( \begin{aligned} &2\alpha \left(\sin\left(\frac{\pi}{2}\right)\right)^2 (r[s])^2 \frac{dr}{ds} \left(\frac{d\varphi}{ds}\right)^2 \\ &+ (r[s])^3 (r[s] - 2) \frac{dr}{ds} \frac{dt}{ds} - 2\alpha ((r[s]) - 2) \left(\frac{dt}{ds}\right)^2 \end{aligned} \right) \left(\sin\left(\frac{\pi}{2}\right)\right)^2}{2\alpha \left(\sin\left(\frac{\pi}{2}\right)\right)^2 r[s] \frac{d\varphi}{ds} + \left((r[s])^2 - 2r[s]\right) \frac{dt}{ds}} \\
R_1 &:= c_3 d_3 p_1 - d_3 p_3 c_1 - p_4 d_1 c_3 \\
R_2 &:= p_3 \left( c_1 Q_1 - B - \hat{E} \right) \\
R_3 &:= p_4 \left( d_1 Q_1 - F + \hat{J} \right) \\
R_4 &:= -p_1 Q_1
\end{aligned}$$

### 2.3 MPD Equations for spinless test particle in a weak Kerr metric

The traditional form of MP equations is [18]

$$\frac{D}{ds} \left( m u^\lambda + u_\lambda \frac{DS^{\lambda\mu}}{ds} \right) = -\frac{1}{2} u^\pi S^{\rho\sigma} R_{\pi\rho\sigma}^\lambda \quad (32)$$

First of all, we consider the case of the motion of a spinning test particle in equatorial circular orbits ( $\theta = \pi/2$ ) from the weak Kerr source, that is,  $a/r \ll 1$  and  $MG/c^2$ . For this case we take [26]

$$u^1 = 0, \quad u^2 = 0, \quad u^3 = \text{const}, \quad u^4 = \text{const} \quad (33)$$

when the spin is perpendicular to this plane and the MP condition (3), with

$$S_1 \equiv S_r = 0, \quad S_2 \equiv S_\theta \neq 0, \quad S_3 \equiv S_\varphi = 0. \quad (34)$$



The equation is given by

$$\begin{aligned}
& -y_1^3 * y_7^2 - 2\alpha * y_7 y_8 + y_8^2 - 3 * \alpha * \varepsilon_0 y_7^2 + 3\varepsilon_0 y_7 y_8 - 3\alpha \varepsilon_0 y_8^2 y_1^{-2} + 3\alpha \varepsilon_0 y_1^2 y_7^4 \\
& -\alpha \varepsilon_0 \left(1 - \frac{2}{y_1}\right) y_8^4 y_1^{-3} + \varepsilon_0 (y_1^6 - 3y_1^5) y_7^3 y_8 y_1^{-3} + \alpha \varepsilon_0 (3y_1^3 - 11y_1^2) y_7^2 y_8^2 y_1^{-3} \\
& + \varepsilon_0 (-y_1^3 + 3y_1^2) y_7 y_8^3 y_1^{-3} = 0
\end{aligned} \tag{35}$$

Then, for the case when the particle does not have spin the set of equations (35) with the dimensionless quantities  $y_i$  (8) and (9) is reduced to

$$-y_1^3 * y_7^2 - 2\alpha * y_7 y_8 + y_8^2 = 0 \tag{36}$$

where  $\alpha = a/M$ .

In addition to Eq. (36), we take the condition  $u_\mu u^\mu = 1$  and obtain

$$-y_1^2 * y_7^2 + 4\alpha \frac{y_7 y_8}{y_1} + \left(1 - \frac{2M}{y_1}\right) y_8^2 = 1 \tag{37}$$

We solve the system of equations (36) and (37) for the case of a circular orbit and obtain the values of  $y_7 = Mu^3$  and  $y_8 = u^4$ .

## 2.4 Constants of motion for a weak Kerr metric

With the Tulczyjew-Dixon condition (4), we determine the center of mass of the particle and let  $u^\mu$  be its four-velocity; also, the MPD equations (1) and (2) yield a unique  $u^\mu$ , namely [27]

$$u^\mu = V^\mu + \frac{1}{2} \left( \frac{S^{\mu\nu} R_{\nu\rho\sigma\kappa} V^\rho S^{\sigma\kappa}}{m^2 + \frac{1}{4} R_{\chi\xi\zeta\eta} S^{\chi\xi} S^{\zeta\eta}} \right). \tag{38}$$

$u^\mu$  and  $V^\mu$  are named by Dixon as kinematical four velocity and dynamical four velocity, respectively [23].

Since  $p_\mu p^\mu = \text{constant}$  and  $S_{\rho\sigma} S^{\rho\sigma} = \text{constant}$  along the particle trajectory [28], we may set

$$u_\mu u^\mu = -1, \quad S_0^2 = S_\mu S^\mu = \frac{1}{2m^2} S_{\mu\nu} S^{\mu\nu}, \tag{39}$$

and with these expressions, we obtain the center of mass condition and the relation between the spin tensor and the vector spin.

Next, we reduced the set of MPD equations given by Plyatsko *et al.* [6] for the case when a spinning test particle is a weak Kerr metric in the equatorial plane and follows a circular orbit. Then, for the initial conditions we need the values of both the energy ( $E$ ) and the component  $z$  of the angular momentum ( $J$ ) for a weak Kerr metric. In this case, the constants of motion are given by

$$E = m(g_{44} + g_{34})V^4 + \frac{Ma}{r^2}S^{13} - \frac{M}{r^2} \frac{g_{33}V^3}{g_{44}V^4}S^{13} \quad (40)$$

$$J_z = -m(g_{33} + g_{34})V^3 + r \sin^2 \theta S^{13} - \frac{Ma}{r^2} \frac{(g_{33} + g_{34})V^3}{(g_{44} + g_{34})V^4}S^{13} \quad (41)$$

where  $V^3$  and  $V^4$  are components of the dynamical 4-velocity and  $S^{13}$  is the perpendicular component of the spin vector.

We yield the components of the 4-velocity  $u^\lambda$  (38) for the case of a spinning test particle in a weak Kerr metric and in the equatorial plane  $\theta = \pi/2$  when spin is orthogonal to this plane and has a constant radius ( $x^1 = r = \text{constant}$ ). We use the Boyer-Lindquist coordinates ( $x^1 = r$ ,  $x^2 = \theta$ ,  $x^3 = \varphi$ ,  $x^4 = t$ ). We have

$$u^1 = 0, \quad u^2 = 0, \quad u^3 \neq 0, \quad u^4 \neq 0, \quad (42)$$

$$S^{12} = 0, \quad S^{23} = 0, \quad S^{13} \neq 0 \quad (43)$$

In addition to (43) by Tulczyjew-Dixon condition (4) we write

$$S^{14} = -\frac{P_3}{P_4}S^{13}, \quad S^{24} = 0, \quad S^{34} = \frac{P_1}{P_4}S^{13} \quad (44)$$

Using (39), (42)-(44) and taking the components of the Riemann tensor for the weak Kerr metric in the equatorial plane, from (38) we obtain

$$\begin{aligned} u^1 &= NV^1 \left( 1 + \frac{3M}{r^3} V_3 V^3 \frac{S_0^2}{m^2 \Delta} + a \frac{S_0^2}{m^2 \Delta} k_1 \frac{V_3}{V^4} \right) \\ u^2 &= V^2 = 0 \\ u^3 &= NV^3 \left( 1 + \frac{3M}{r^3} (V_3 V^3 - 1) \frac{S_0^2 M}{m^2 \Delta} + a \frac{S_0^2 M}{m^2 \Delta} k_3 \frac{V^3}{V^4} \right) \\ u^4 &= NV^4 \left( 1 + \frac{3M}{r^3} V_3 V^3 \frac{S_0^2}{m^2 \Delta} + a \frac{S_0^2 M}{m^2 \Delta} k_4 \left( \frac{V^3}{V^4} \right)^2 \right) \end{aligned} \quad (45)$$

where the constants  $k_1$ ,  $k_3$  and  $k_4$  are given by

$$\begin{aligned} k_1 &= \frac{3M \left( 1 - \frac{4M}{3r} \right)}{g_{11}g_{44}} \\ k_3 &= k_1 \\ k_4 &= \frac{k_1}{rg_{44}} \end{aligned}$$

and the expression  $\Delta = 1 + \frac{1}{4m^2} R_{\chi\xi\zeta\eta} S^{\chi\xi} S^{\zeta\eta}$  for the weak Kerr metric is given by

$$\Delta = 1 + \frac{S_0^2 M}{m^2 r^3} \left( 1 - 3V_3 V^3 - aA \frac{V^3}{V^4} \right) \quad (46)$$

where  $a = J/Mc$  is the angular density of the central mass and

$$A = \frac{3M \left(1 - \frac{4M}{3r}\right) g_{33}}{g_{44}}$$

We insert (46) into (45), we get

$$\begin{aligned} u^1 &= \frac{NV^1}{\Delta} \left( 1 + \frac{S_0^2 M}{m^2 r^3} - a \frac{S_0^2}{m^2} (MA - k_1) \frac{V^3}{V^4} \right) \\ u^3 &= \frac{NV^3}{\Delta} \left( 1 - \frac{2S_0^2 M}{m^2 r^3} - a \frac{S_0^2}{m^2} (MA - k_3) \frac{V^3}{V^4} \right) \\ u^4 &= \frac{NV^4}{\Delta} \left( 1 + \frac{2S_0^2 M}{m^2 r^3} - a \frac{S_0^2}{m^2} \left( MA - k_4 \left( \frac{V^3}{V^4} \right)^2 \right) \frac{V^3}{V^4} \right) \end{aligned} \quad (47)$$

We introduce

$$\varepsilon = \frac{|S_0|}{mr} \quad (48)$$

and obtain the expression for  $N$  from the conditions

$$\begin{aligned} V_\lambda V^\lambda &= 0 \\ V_\lambda S^{\lambda\nu} &= 0 \end{aligned}$$

$N$  is given by

$$N = \frac{\Delta}{R} \quad (49)$$

where

$$R = \left( m^2 \Delta^2 + S_0^4 R^{\mu\tau\rho\delta} R_{\mu\tau\rho\delta} \right)^{\frac{1}{2}} \quad (50)$$

We insert (49) into (47) and obtain the components from  $V^\lambda$

$$\begin{aligned} V^1 &= \frac{Ru^1}{\left( 1 + \frac{S_0^2 M}{m^2 r^3} (1 + a(A + k_1)) \right)} \\ V^3 &= \frac{Ru^3}{\left( 1 - \frac{S_0^2 M}{m^2 r^3} (1 + a(A - k_3)) \right)} \\ V^4 &= \frac{Ru^4}{\left( 1 + \frac{S_0^2 M}{m^2 r^3} \left( 1 - a \left( A + 3(u^3)^2 k_4 \right) \right) \right)} \end{aligned} \quad (51)$$

We replace the components of the dynamical 4-velocity (51) in the constants of motion (40) and (41) for the case of a spinning test particle in a weak Kerr field.

### 3 Gravitomagnetic clock effect for spinning test particles

For checking our results, we review the papers in regarding to Gravitomagnetic clock effect [29] and compare their numerical results with ours. There is a phenomenon called the gravitomagnetic clock effect which consists of a difference in the time it takes for two test particles to travel around a rotating massive body in the equatorial plane and in opposite directions [2]. This difference is given by  $t_+ - t_- = 4\pi a/c$ , where  $a = J/Mc$  is the angular density of the central mass. Tartaglia has studied the geometrical aspects of this phenomenon [17], [30] and Faruque yields the equation of the gravitomagnetic clock effect with spin as

$$t_+ - t_- = 4\pi a - 6\pi S_0, \quad (52)$$

where  $S_0$  is the magnitude of the spin.

In true units this relation is given by

$$t_+ - t_- = \frac{4\pi J_M}{Mc^2} - \frac{6\pi J}{mc^2}, \quad (53)$$

where the first relation of the right could be used to measure  $J/M$  directly for an astronomical body; in the case of the Earth  $t_+ - t_- \simeq 10^{-7}$  s, while for the Sun  $t_+ - t_- \simeq 10^{-5}$  s [31].

### 4 Numerical comparison for spinless and spinning test particle via MPD equations

In this section we give the numerical results for the case of a spinning satellite orbiting around the Earth [32]. We took the data of the Ariane-5 satellite which is a space European vehicle that is part of the Ariane family [33]. For our calculations we assume the satellite follows a circular orbit with radius equal to  $3.5 \times 10^6$  m and travels in the equatorial plane. Of course, the satellite has an orbit that is more complex. The initial conditions are given by geometrized units, where the gravitational constant  $G$ , and speed of light  $c$ , are set equal to one.

According with the features of the Ariane-5 satellite, its initial conditions are

$$\text{Mass}(m)_{\text{satellite}} = 3 \times 10^3 \text{ kg}, \quad \varepsilon_0 = \frac{S_0}{mr} = 6.0976 \times 10^{-11}$$

The fundamental frequency in the longitudinal axis equals 30 Hz and the components of the four velocity of the satellite are given by a set of equations (35) and (37) for a orbit of  $3.5 \times 10^6$  m. For this case, in ordinary units, the azimuthal component is  $u^3 = 2.42294 \times 10^3$  m/s.

We take the set of MPD equations for a spinning test particle in a weak Kerr metric (Appendix A) and write the initial conditions for this satellite. With the Runge-Kutta method of order 4 [34], we obtain the cartesian coordinates for

a circular orbit when the satellite orbits in the same sense of rotation of the central source ( $a$ ). The program code is in the Appendix B. We register the time that satellite takes for doing a lap. Then we take the same data in the case when the satellite orbits with the sense of rotation contrary to that of the central source ( $a$ ). Finally, we take the difference of time in these two orbits and obtain

$$\Delta\tau_{\text{spinning}} = \tau_+ - \tau_- = 7.275957 \times 10^{-7} \text{ s} \quad (54)$$

Now we take the case when the test particle does not have spin and calculate the cartesian coordinates ( $x, y, z$ ) for a circular orbit of a spinless test particle around a rotating body mass both in the same sense of rotation of the central mass and in opposite direction. In this case, we take the set of MPD equations for a spinless particle, Eqs. (36) - (37). There is a spinless particle in the equatorial plane ( $u^2 = 0$ ) and with a radius constant ( $u^1 = 0$ ). We assume the same initial conditions as in the previous case. As the above part, we calculate the difference of time of two particles travel in the same orbit, but in opposite directions, and the result is

$$\Delta\tau_{\text{spinless}} = \tau_+ - \tau_- = 9.01062 \times 10^{-7} \text{ s} \quad (55)$$

This result is according to the literature. In some papers this difference of time is called Effect Gravitomagnetic and is given by the expression [32]

$$(\tau_+ - \tau_-)_{\phi=2\pi} \simeq 4\pi \frac{J_{\oplus}}{M_{\oplus}c^2} \simeq 10^{-7} \text{ s} \quad (56)$$

where  $M_{\oplus}$  and  $J_{\oplus}$  are the values of the mass of Earth and the angular momentum respect

According to the results, the spinless test particle in a positive sense completes a full orbit before the particle with the sense of rotation contrary. This phenomenon is due to drag of the inertial frames with respect to infinity and is called the Lense-Thirring effect [35]. In the case of the spinning test particles, not only there is a difference in the time given by the Lense-Thirring effect, but also by a coupling between the angular momentum of the central body with the spin of the particle [36]. The features change if the test particle rotates in one direction or the other; therefore, the period is different for one sense and for the other, and if the particle has spin or not. The difference of time between the spinless particles and the spinning particles is so small that the result is the same order of the shift ( $10^{-7}$  s). In other words, when the spinning test particle is very small compared with the central mass, the influence of the value of spin in the shift of time is insignificant in regard to lapse of time.

## 5 Conclusions

In this paper, we take the Mathisson-Papapetrou-Dixon (MPD) equations given by Plyatsko *et al.* and obtained explicitly the MPD equations for the case when the spinning test particle is orbiting in a rotating weak field. Work that was

not in the literature. In addition, we gave a scheme for the eleven equations of the full set of equations of motion when the particle is orbiting a rotating gravitational field. In the second part, we worked the constants of motion such as the energy ( $E$ ) and the angular momentum ( $J_z$ ) of the spinning test particle in a weak Kerr metric. Finally, we calculated the trajectories in cartesian coordinates ( $x, y, z$ ) of test particles both spinless and spinning orbiting in a weak Kerr metric and compared the time of two circular orbits in the equatorial plane for two test particles that travel in the same orbit but in opposite directions. In the case of the Earth, both for the spinless particles and the spinning particles there is a difference of time in their trajectories when they describe a full revolution with respect to an asymptotically inertial observer. This phenomenon is called Gravitomagnetic Effect. From this situation, we concluded that this shift, in the case of the spinless test particles, is given by the angular momentum from the central source which drags the inertial systems in the same sense of the rotation of the rotating massive body. For the case of the spinning test particles, this time lapse is given not only by the angular momentum from the central mass, but also by the couple between the angular momentum from the massive rotating body and the parallel component of the spin of the test particle. In the MPD equations, this couple is given by the relationship between the components of the Riemman tensor ( $R^\mu{}_{\nu\rho\sigma}$ ) and the spin tensor ( $S^{\rho\sigma}$ ).

In the future we will work in the set of Equations of motion of a test particle both spinless and spinning for spherical orbits, that is, with constant radius and out of the equatorial plane in a weak Kerr metric. In addition, we are interesting in relating these equations with the experiments type Michelson and Morley.

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## Appendix A Weak Kerr Metric

The components of a weak Kerr Metric are given by

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2M}{r} & 0 & 0 & \frac{2Ma \sin^2 \theta}{r} \\ 0 & -1 - \frac{2M}{r} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ \frac{2Ma \sin^2 \theta}{r} & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}$$

## Appendix B Program code

```

TF = 1 * 106;  $\epsilon_0 = 6.0976 * 10^{-11}$ ;  $y1 = 3.1085 * 10^4$ ;  $y10 = y1 * \epsilon_0$ ;
y5 = 0;  $\alpha = 1.9765 * 10^{-16}$ ;
M = 4.431948 * 10-3;  $m = 2.228 * 10^{-24}$ ;  $y2 = \frac{\pi}{2}$ ;
SetPrecision[NSolve[ $\{-(y1)^3(y7)^2 - 2 * \alpha * y7y8 + (y8)^2 - 3 * \alpha * \epsilon_0(y1)^2 * \frac{(y7)^2}{(y1)^2} + 3 * \epsilon_0 * y7y8 - 3 * \alpha * \epsilon_0 * \frac{(y8)^2}{(y1)^2} +$ 
 $3 * \alpha * \epsilon_0 * (y1)^2(y7)^4 - \alpha * \epsilon_0 * (1 - \frac{2}{y1})\frac{(y8)^4}{(y1)^3} + \epsilon_0((y1)^6 - 3(y1)^5)\frac{(y7)^3y8}{(y1)^3} + \alpha * \epsilon_0(3(y1)^3 - 11(y1)^2) * \frac{(y7)^2(y8)^2}{(y1)^3} +$ 
 $\epsilon_0 * (-(y1)^3 + 3(y1)^2)\frac{y7(y8)^3}{(y1)^3} == 0, -(y1)^2(y7)^2 + \frac{4 * \alpha * y7 * y8}{y1}$ 
 $+ (1 - \frac{2}{y1})(y8)^2 == 1\}$ , {y7, y8}], 10]
system1 = {y3[s] == y7, y4[s] == y8, y3[0] == 0, y4[0] == 0};
sol1 = NDSolve[system1, {y3, y4, y3', y4'}, {s, 0, TF},
Method -> "Automatic", MaxSteps -> 1 * 1010]
graph1 = ParametricPlot3D[Evaluate[{(y1) * sin  $\frac{\pi}{2}$  * cos[y3[s]], (y1) * sin  $\frac{\pi}{2}$  *
sin[y3[s]], (y1) * cos  $\frac{\pi}{2}$ }/.sol1], {s, 0, TF},
AxesLabel -> {"x", "y", "z"}, PlotStyle -> {Blue}]
SetPrecision[Table[{s, y1 * sin[y2] * sin[y3[s]]}/.sol1, {s, 0, TF, 9.53674316406 *
10-7}], 20]

```

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